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LETTER TO THE EDITOR

Stability of the Sherrington–Kirkpatrick solution of a spin glass model: a reply

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Abstract. De Almeida and Thouless have argued that the stationary point used by Sherrington and Kirkpatrick (SK) in their evaluation of the free energy of a spin glass is unstable in the ordered phase. We demonstrate an inconsistency in their argument and show that there is a systematic justification of the SK procedure. The fundamental error in the complete SK analysis remains unresolved.

In a recent paper de Almeida and Thouless (1978 to be referred to as AT) have argued that a stationary point employed by Sherrington and Kirkpatrick (1975) in their study of an infinite-ranged spin glass model is unstable. In this Letter we point out that AT's criticism of SK is itself open to question and, further, we expose the contrary reasoning underlying the SK solution. For simplicity we restrict ourselves here to the case of zero mean exchange and zero external field. The reader is referred to the original papers for background detail.

The relevant question concerns the minimisation of an effective free energy function

$$f = kT\left[\sum_{(\alpha\beta)} \tfrac{1}{2} y^{(\alpha\beta)2} - \ln \mathrm{Tr} \exp\left((J/kT) \sum_{(\alpha\beta)} y^{(\alpha\beta)} S^\alpha S^\beta\right)\right] \qquad (1)$$

where the indices α, β run from 1 to n, $(\alpha\beta)$ denotes distinct pairs of indices with $\alpha \neq \beta$, the spins S^α are Ising taking the values ± 1, and the trace is over all 2^n values of S^α, and, further, the analytic continuation of the results to $n \to 0$. SK took as the relevant extremum that with all the $y^{(\alpha\beta)}$ equal and it is this that AT contest. The basis of the contestation lies in an examination of the deviation of f from its value with all $y^{(\alpha\beta)}$ equal; for a stable solution all the eigenvalues of the quadratic deviation must be positive definite. AT argued that for temperatures less than (J/k) one of the eigenvalues becomes negative in the $n \to 0$ limit and thus the SK choice is unstable.

As de Almeida and Thouless have shown, for large integral n there are only three different eigenvalues

$$\begin{aligned}
\lambda_1 &= 1 - (J/kT)^2[1 + \tfrac{1}{2}(n-2)(n-3)\langle S^\alpha S^\beta S^\gamma S^\delta\rangle + 2(n-2)\langle S^\alpha S^\beta\rangle - \tfrac{1}{2}n(n-1)\langle S^\alpha S^\beta\rangle^2] \\
\lambda_2 &= 1 - (J/kT)^2[1 - (n-3)\langle S^\alpha S^\beta S^\gamma S^\delta\rangle + (n-4)\langle S^\alpha S^\beta\rangle] \qquad (2) \\
\lambda_3 &= 1 - (J/kT)^2(1 + \langle S^\alpha S^\beta S^\gamma S^\delta\rangle - 2\langle S^\alpha S^\beta\rangle)
\end{aligned}$$

where different Greek letters refer to different indices. The correlation functions are defined by

$$\langle S^\alpha \dots S^\omega \rangle = \frac{\mathrm{Tr}\, S^\alpha \dots S^\omega \exp[(J/kT)y\Sigma_{(\mu\nu)}S^\mu S^\nu]}{\mathrm{Tr}\exp[(J/kT)y\Sigma_{(\mu\nu)}S^\mu S^\nu]}. \tag{3}$$

The degeneracies of the eigenmodes are

$$g_1 = 1, \qquad g_2 = (n-1), \qquad g_3 = \tfrac{1}{2}n(n-3). \tag{4}$$

The self-consistency equation for y is

$$(kTy/J) = \frac{\mathrm{Tr}\, S^\alpha S^\beta \exp[(J/kT)y\Sigma_{(\mu\nu)}S^\mu S^\nu]}{\mathrm{Tr}\exp[(J/kT)y\Sigma_{(\mu\nu)}S^\mu S^\nu]}. \tag{5}$$

Using an analytic continuation of (3) due to Edwards and Anderson (1975, see also SK), de Almeida and Thouless studied the eigenvalues (2) only in the limit $n \to 0$. They observed that in this limit λ_3 apparently becomes negative below the temperature (J/k). In fact, however, it is physically apparent, and also borne out by (4), that the three separate eigenvalues are meaningful only for integral $n \geq 3$. To illustrate the appearance of spurious results it suffices to consider $n = 2$, for which (1) corresponds to a perfectly physical system. It is obvious from the fact that in this case there is but one $(\alpha\beta)$ combination that only one eigenmode is meaningful; that is λ_1. Analytic continuation of λ_3 to $n = 2$ is clearly spurious; this is further evident from the 'cancelling' degeneracies of the now-identical eigenvalues λ_2 and λ_3. Similarly AT's analytic continuation of λ_3 to $n \to 0$ is without any apparent physical significance†. It seems at least as appropriate to consider the stability of the solution with all $y^{(\alpha\beta)}$ equal for the physically sensible cases of integral $n \geq 3$ and extrapolate only the conclusion. This conclusion is that the eigenvalues are all non-negative and the solutions stable. There is however a subtlety in that the transitions are all first order for $n > 2$.

For $n = 2$ equation (5) is that of a pure Ising model treated in mean field approximation and yields a second-order transition at $T = J/k$. For $n > 2$, however, the transition is first order‡ with $T_c > J/k$. This is illustrated graphically in figure 1 where is plotted the right-hand side of (5) for $n = 4$; we use a notation $x = Jy/kT$ and call the RHS of (5) $g(x)$; the left-hand side is simply $(J/kT)^{-2}x$. The relevant point indicative of the necessity of a first-order solution is the finite curvature at $x = 0$. This is apparent for general n from the small-x expansion of $g(x)$,

$$g(x) = x + (n-2)x^2 + \dots. \tag{8}$$

Only for $n = 2$ does $g(x)$ have the zero curvature that is the signature of a possible second-order transition. To further illustrate the point there is also shown in figure 1 the LHS of (5) for the temperature corresponding to the phase transition; the transition temperature for $n = 4$ occurs at $T_c \simeq 1{\cdot}27J/k$, $x_c \simeq 0{\cdot}395$, $q_c \simeq 0{\cdot}638$.

† This should be contrasted with the analytic continuation of $\langle S^\alpha S^\beta \rangle$ to $n \to 0$ (as employed by SK) whose physical significance lies in that

$$\lim_{n\to 0} \langle S^\alpha S^\beta \rangle_{\alpha \neq \beta} = \overline{|\langle S_i \rangle|^2}$$

where the bar refers to a spatial average.

‡ This may be considered as due to the quasi-Potts character of the $n > 2$ models.

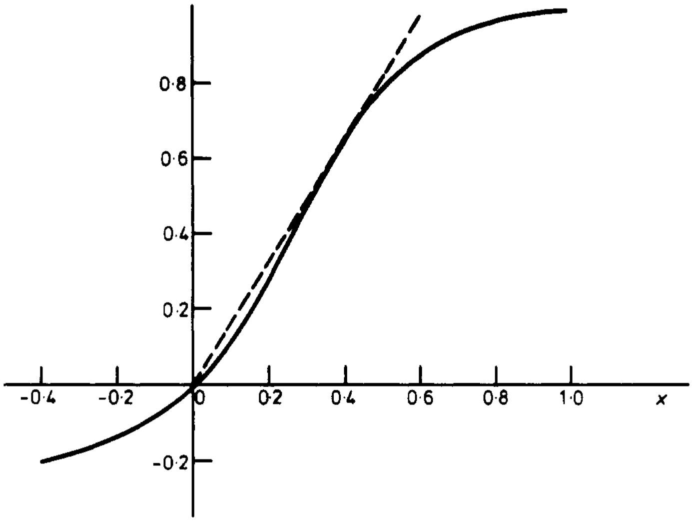

Figure 1. Curves relevant to transcendental solution for $n=4$. Full curve: $g(x)$. Broken curve: $(J/kT)^2x$ for $T=T_c$.

It is straightforward to show that λ_2 and λ_3 are positive for integral $n \geqslant 3$ in the physical regions. Figure 2 depicts these eigenvalues for $n=4$ as a function of x in the ordered region, $T<T_c$. For $T>T_c$ all the correlation functions of different S^α are identically zero so that λ_1 and λ_2 are both $[1-(J/kT)^2]$, which is positive since $T_c>J$. There is thus no instability associated with λ_2 or λ_3. In the ordered phase λ_1 may be simply expressed in terms of $g(x)$ (for all n) as

$$\lambda_1 = 1 - \left(\frac{\partial g(x)}{\partial x}\bigg/\frac{\partial g(x)}{\partial x}\bigg|_{x_c}\right) \tag{9}$$

where x is given by the solution to (5) and x_c is the value at the phase transition. For $T>T_c$ λ_1 is $|1-(J/kT)^2|$. It is thus clear that λ_1 also is always positive except exactly at $x=x_c$ where it becomes zero signalling the transition. We thus conclude that SK's choice of all $y^{(\alpha\beta)}$ equal can be justified. The analytic continuation of (5) to $n\to 0$ (via

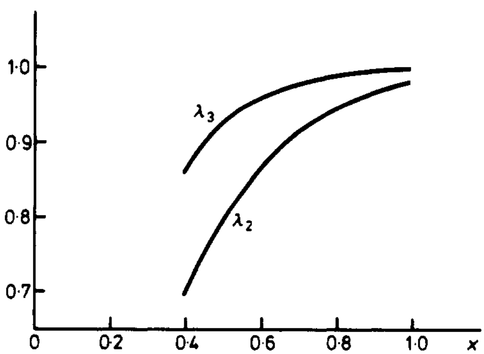

Figure 2. Eigenvalues λ_2, λ_3 in the ordered phase for $n=4$ as a function of the parameter x. In the disordered phase both are $[1-(J/kT)^2]$.

the Edwards–Anderson parenthetisation) remains, however, non-trivial; it is clear that the transition temperature is not simply continuable and also the physical interpretation of $\lim_{n\to 0} q$ as $\overline{|\langle S_i\rangle|^2}$ in the spin glass case implies that for this application q must be restricted to positive values.

More details of n-replicated models will be discussed elsewhere.

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